



8.5 The range of a linear map

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Introduction

- In this topic, we will
 - Define the range of a linear map
 - Consider some consequences
 - Look at finite-dimensional maps
 - Consider the advance and delay operators





Describing polynomials

- Recall that a polynomial of degree n is a collection of $n + 1$ coefficients
 - What is the significance of these coefficients?
 - Can changing one number completely change the properties of that polynomial?
 - Short answer, yes...
 - What you did in secondary school is determine the important characteristics of polynomials:
 - The roots of a polynomial
 - The coefficient of the leading term





Describing linear maps

- An $m \times n$ matrix is a collection of mn entries
 - What is the significance of this?
 - Can changing one number completely change the properties of that matrix?
 - Short answer, yes...
 - What we want to do is describe the useful characteristics of linear maps
 - These characteristics help understand what a linear map does, and how to design linear maps
 - The first two such features are to describe the *range* and *null space* of that linear map
 - We will start with the range





The zero map and operator

- Given a linear map $A: \mathcal{U} \rightarrow \mathcal{V}$, the range is the collection of all vectors in $\mathcal{V}$ such that at least one vector in $\mathcal{U}$ is mapped to that vector.
- That is, $\text{range}(A)$ equals all vectors $A\mathbf{u}$ where $\mathbf{u}$ is a vector in $\mathcal{U}$
 - You will note that $\text{range}(A) \subseteq \mathcal{V}$
 - Also, $\text{range}(A)$ always has at least one element: $\mathbf{0}_{\mathcal{V}}$





The range is a subspace

Theorem

If $A: \mathcal{U} \rightarrow \mathcal{V}$ is a linear map, the range of A is a subspace of $\mathcal{V}$.

Proof:

If $\mathbf{v}_1, \mathbf{v}_2 \in \text{range}(A)$, then we must show $\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 \in \text{range}(A)$

If $\mathbf{v}_1, \mathbf{v}_2 \in \text{range}(A)$, then there must be $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$

such that $A\mathbf{u}_1 = \mathbf{v}_1$ and $A\mathbf{u}_2 = \mathbf{v}_2$

Now, because $\mathcal{U}$ is a vector space, $\alpha \mathbf{u}_1 + \beta \mathbf{u}_2 \in \mathcal{U}$

But because A is linear, $A(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) = \alpha A\mathbf{u}_1 + \beta A\mathbf{u}_2$
 $= \alpha \mathbf{v}_1 + \beta \mathbf{v}_2$

So since $\alpha \mathbf{u}_1 + \beta \mathbf{u}_2 \in \mathcal{U}$, it follows

$$A(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 \in \text{range}(A)$$

Thus, the range is closed under vector addition
and scalar multiplication, so it is a subspace. ■





The image of a subspace is a subspace

Theorem

If S is a subspace of $\mathcal{U}$ and $A: \mathcal{U} \rightarrow \mathcal{V}$ is a linear map, the image of S , AS is a subspace of $\mathcal{V}$.

Proof:

The proof is similar to the previous proof.





The range of the zero map

Theorem

If $A: \mathcal{U} \rightarrow \mathcal{V}$ is a linear map,
 $\text{range}(A) = \{\mathbf{0}_{\mathcal{V}}\}$ if and only if $A = O$

Proof

If $\text{range}(A) = \{\mathbf{0}_{\mathcal{V}}\}$, then $A\mathbf{u} = \mathbf{0}_{\mathcal{V}}$ for all vectors $\mathbf{u}$

However, the zero map O is that map such that

$$\mathbf{u} = \mathbf{0}_{\mathcal{V}} \text{ for all vectors } \mathbf{u}$$

Two maps are equal if $A\mathbf{u} = O\mathbf{u}$ for all $\mathbf{u}$,

so the range is just the zero vector if and only if
the map is the zero map. ■





Definition: onto

Definition

If $A: \mathcal{U} \rightarrow \mathcal{V}$ and $\text{range}(A) = \mathcal{V}$, we say that A is *onto*.

It is also said that A is *surjective*.

– Recall that

- A *surcharge* is a charge on top of what you're already paying for a product
- A *surplus* is over and above what you need
- Thus, a map is surjective if every point in the domain is mapped onto





Definition: onto

- What is “onto”?
 - Any odd polynomial with real coefficients of a real variable maps onto the reals
 - No even polynomial with real coefficients of a real variable maps onto the reals
 - Such a polynomial always has either a global minimum or a global maximum
 - For $t > 0$, $\ln(t)$ maps onto the real numbers
 - The range of e^t for real t is the open interval $(0, \infty)$, so therefore it is not onto
 - On the other hand, $e^t \sin(t)$ is onto the reals
- Challenge: prove that every non-constant polynomial with complex coefficients maps onto $\mathbb{C}$





Range of finite-dimensional linear mappings

- If $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$, the range of A equals the span of the column vectors
 - That is, if $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$, then $\text{range}(A) = \text{span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$
 - You can use Gaussian elimination to find a basis for the range
- For example,

$$A = \begin{pmatrix} 1 & 2 & -1 & -3 & -2 \\ 3 & 8 & 2 & -11 & -2 \\ 4 & 14 & 11 & -18 & 7 \\ -2 & -8 & -8 & 10 & -10 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & -3 & -2 \\ 0 & 2 & 5 & -2 & 4 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Thus, a basis is Columns 1, 2 and 5 of the matrix A
- The range is a 3-dimensional subspace of $\mathbf{R}^4$





Range of finite-dimensional linear mappings

- If $A: \mathbf{F}^n \rightarrow \mathbf{F}^m$,
 - The dimension of $\text{range}(A)$ is $\text{rank}(A)$
 - A is onto if $\text{rank}(A) = m$





Operators on discrete-time signals

- Recall the delay and advance operators on discrete-time signals:
- If $x = (x[0], x[1], x[2], x[3], \dots)$,
 $Dx = (0, x[0], x[1], x[2], \dots)$ and the range is all signals x with $x[0] = 0$
 $Ex = (x[1], x[2], x[3], x[4], \dots)$ and the range is all signals





Summary

- Following this topic, you now
 - Know the definition of the range
 - Know that the range of a linear map is always a subspace
 - Have become familiar with some additional results
 - Understand the definition of a map being onto
 - Know how to determine the range of a finite-dimensional linear map





References

- [1] https://en.wikipedia.org/wiki/Range_of_a_function





Acknowledgments

None so far.





Colophon

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The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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